Back Paper | ISI Bangalore | November 2019

Algebra I | 100 Points

Notes.

(a) Justify all your steps. You may use any result proved in class unless you have been asked to prove the same.

(b) $\mathbb{N}$ = natural numbers, $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers.

(c) $\mathbb{Z}_n$ = the set of integers modulo n, $(\mathbb{Z}_n)^\times$ = the set of those integers in $\mathbb{Z}_n$ that are coprime to n.

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1. [15 points] Let G be a group.
   (i) Define what it means for a subgroup H to be normal in G.
   (ii) Give an example of G and a subgroup H such that H is not normal in G.
   (iii) Let N be a normal subgroup of G. Define a group operation on the set of all left cosets of N in G. (You must verify that the operation is well-defined and briefly justify why it satisfies the axioms of a group.)

2. [15 points] Prove that the following groups are not isomorphic to each other.

   (a) $(\mathbb{Z}, +)$ (b) $(\mathbb{Q}, +)$ (c) $(\mathbb{R}, +)$

3. [10 points] Let G be a group. Let a, b be elements of G. Prove that if ab has finite order then so does ba and in that case, both of them have the same order.

4. [15 points] Let G be a finite group. Consider the function $\phi\colon G \to G$ given by $\phi(x) = x^2$. Prove that ϕ is an isomorphism if and only if G is abelian with every element of G having odd order.

5. [15 points] List all the conjugacy classes of elements in the permutation group S_5. In order to list the classes, you must exhibit one element from each class and also indicate how many elements are there in each class.

6. [15 points] Let p be a prime. Define what it means for a finite group to be called a p-group. Prove using the class equation that the center of a (finite) p-group always has more than one element. Finally, prove that a group of order p^2 is always abelian.

7. [15 points] Prove that any group of order 15 is cyclic.